Polynomials (Ch.1) Study Guide by BS, JL, AZ, CC, SH, HL

- **Lagrange Interpolation is a method of fitting an equation to a set of points that functions well when there are few points given.**
- **Sasha's method is a sweet alternative to the lagrange interpolation and uses the formula f(x) = A + B(x-1) + C(x-1)(x-2) + D(x-1)(x-2)(x-3) and so on to find a function that fits an input output table.**
- **The end behavior of a graph and its zeroes can be determined based on its equation**.
- **A graph's zeros and end behavior tell what its equation is.**
- **Factoring polynomials is the inverse process of multiplying polynomials.**
- **Two ways to divide polynomials are fractional division and long division.**

<u>List</u>

- Theorems:
  - **Factor Theorem**: Suppose f(x) is a polynomial. Then the number a is a root of the equation f(x)=0 if and only if x-a is a factor of f(x).
  - **Remainder Theorem:** If you divide a polynomial f(x) by x-a, where a is a number, the remainder is f(a).
- Corollaries:
  - A polynomial of degree n can have at most n distinct real number zeros.
  - If 2 polynomials of degree n agree at n+1 inputs, they are identical.
- Properties:
  - **Euclidean Property for polynomials:** Given polynomials f(x) and g(x), there are unique polynomials q(x) (the quotient) and r(x) (remainder) such that
    - 1. f(x) = g(x) * q(x) + r(x)
    - 2. deg(r(x)) < deg (g(x))
- Difference of cubes: x^3 - y^3 = (x-y) (x^2 + xy + y^2)
- Sum of cubes: x^3 +y^3 = (x+y)(x^2 - xy + y^2)
- Vocabulary
  - **Polynomial:** monomial or sum of two or more monomials
  - **Degree of a polynomial:** greatest degree among all the monomials in the polynomial (linear, quadratic, cubic, quartic, quintic)
  - **End behavior:** if the graph goes up or down on its end parts and if its ends go the same way or not
  - **Zero/Root:** a solution to a function, or where the graph touches the x axis

Lagrange Interpolation is a method of fitting an equation to a set of points that functions well when there are few points given. If given a set of points 3,7 5,3 10,11 then the first step is to set up the equation $f(x)=a(x-5)(x-10)+b(x-3)(x-10)+c(x-3)(x-5)$. At $x=3$, only the $a(x-5)(x-10)$ portion of the equation will not be equal to zero. a is then set so that when $x=3$, $a(x-5)(x-10)=7$. In this case, $a=.5$. This process is repeated for all other multiplication groups. In this case $b=-.3$ and $c=.31429$. The Lagrange interpolation equation of points 3,7 5,3 and 10,11 would, therefore, be $f(x)=.5(x-5)(x-10)-.3(x-3)(x-10)+.31429(x-3)(x-5)$. As one could probably tell, this approach is best suited to smaller data sets, as more points causes the length and complexity of the equations to increase rapidly. The method is best used when data sets are small enough and seemingly random enough that it would be incredibly difficult, if not impossible, to accurately fit another line equation to them.

Sasha's method is a sweet alternative to the lagrange interpolation and uses the formula f(x) = A + B(x-1) + C(x-1)(x-2) + D(x-1)(x-2)(x-3) and so on to find a function that fits an input output table. To find the value of A, B, C, D, and so on, all you have to do is find the starting value (first input-output in table) as the A value and make it so that A + B(x-1) = the second value to find B. Repeat the process until all the values have been identified and then simplify to a polynomial function.

Sometimes you see a function that works perfectly well with the table until the final troll term messes everything up, such as the table below. The function 2x+2 fits perfectly with the table until input value 5, which messes everything up by having output value 13. This may seem horrifying, but the solution is simple, use lagrange interpolation! By using the function 2x+2 and adding on 2x+2 + A(x-1)(x-2)(x-3)(x-4) where A gives 13 when the x is five, we get the function for the table! Which is 2.

Input	Output
1	4
2	6
3	8
4	10
5	13 

The end behavior of a graph and its zeroes can be determined based on its equation. In general, both ends of the graph will end the same way if the degree of the equation is even. If the degree of the equation is odd, then the ends of the graph will end in opposite directions. For equations with an even degree, the result of the end (up or down) depends on the leading coefficient. If the leading coefficient is positive, both ends will go up, and if it is negative, both ends will face down. Next, the degree of the equation is odd, and if the leading coefficient is positive, the left end of the graph will decrease as x gets more negative and the right side will increase as x gets larger. However, if the leading coefficient is negative, the left side of the graph will increase as the x coordinate gets more negative and the right end will decrease as x gets larger.

Summary:
a is the leading coefficient, n is the degree

	a>0	a<0
n is odd	down on left, up on right	up on left, down on right
n is even	both up	both down

In addition, by looking at a graph's equation, one can determine its zeroes. There are 3 ways to find the zeroes of a function:

1) Factor the function and set the factors equal to zero. Solve. The zeroes are -2 and -3.

Example: x^2+5x+6 = (x+2)(x+3)

Set x+2 = 0 and x+3 = 0

x = -2 and x = -3

2) For quadratic functions, use the quadratic formula.

For $ax^2 + bx + c = 0$, the value of x is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3) Graphing. The zeroes are located at the intersections of the graph and the x-axis.

A graph's zeros and end behavior tell what its equation is. Zeros tell one where the graph touches the x axis. For example, if the graph touches the x axis at x=3, then in the equation this is shown by (x-3). If the graph touches the x axis at x=-5, then in the equation this is shown by (x+5). If the graph touches the x axis at x=a, that is shown by (x-a) in the equation. Also, the type of contact a graph makes with the x axis influences the degree of the zeros. If the graph passes through the x axis at x=a, then the multiplicity is even. So the zero (x-a) becomes (x-a)^n where n is an even number. If the graph touches the x axis at x=a, then the multiplicity is odd, so the zero (x-a) becomes (x-a)^n where n is an odd number.

Graph passes through x axis at x=4	Example: (x-4)^2
Graph touches x axis at x=-7	Example: (x+7)^5

In addition, the end behavior of the graph will help tell what the degree and leading coefficient of the equation are. See the previous section for an in depth description of end behavior. Below is a summary:

Graph's right end goes up and left end goes down	Odd degree Positive coefficient
Graph's left end goes up and right end goes down	Odd degree Negative coefficient
Graph ends both go upward	Even degree Positive coefficient
Graph ends both go downward	Even degree Negative coefficient

Important things to remember:

- Find all zeros
- Make sure to find the numerical value of the leading coefficient, not just if it's positive or negative

Factoring polynomials is the inverse process of multiplying polynomials. We can factor a polynomial into the product of a number of simpler expressions of the form (ax+b), where a and b are constants. If we divide a polynomial by the factors after factoring the polynomial, the remainder will be zero. Whenever we factor a polynomial always look for the greatest common factor (GCF) then we determine if the resulting polynomial factor can be factored again.

Here are the most common factoring techniques used with polynomials:

If we have any number of terms then we use GCF:

$$a^4b^2 + a^2b^2 - a^3b^2 = a^2b^2(a^2 + 1 - a)$$

If we have two terms then we could use either the difference of two squares, the sum of two cubes or the difference of two cubes:

$$a^2 - b^2 = (a + b)(a - b)$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

If we have three terms then we use either perfect square trinomials or general trinomials:

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$
$$aex^2 + (ad + be)x + bd = (ax + b)(ex + d)$$

Lastly, if we have four or more terms we use grouping:

$$xa + xb + ya + yb = x(a + b) + y(a + b)$$

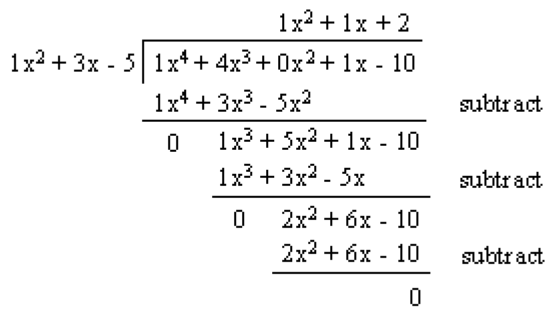
To factor over R, use the quadratic formula. You may get a set of solutions like (-b + √m)/2a and (-b - √m)/2a where m=b^2 - 4ac. The solutions are those plugged into the zero form; as in, (x-(-b+√m)/2a) and (x-(-b-√m)/2a).

Other factoring methods:

- using the quadratic equation - best method when factoring over R
- Z-substitution

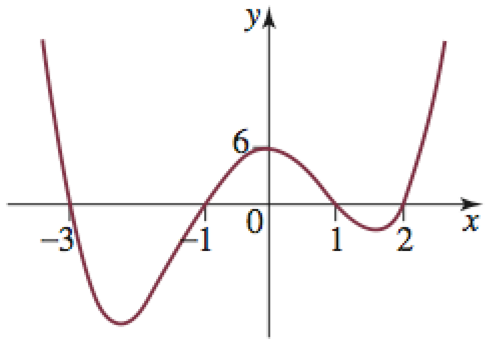
Two ways to divide polynomials are fractional division and long division. The first method, fractional division, is done by factoring both sides of the equation to simplify the problem. This is done by dividing by 1, but substituting something like (x-3)/(x-3) as 1. For example, if there is an equation like (x^2-9)/(x^2-5x+6), you can divide it by (x-3)/(x-3) to simplify it to (x-3)/(x-2). This would get you to a simpler equation until there are no more factors to divide both sides by.

After this, you can move on to the second method, long division. Long division is done the same way as dividing numbers. As seen in the photo below, you go from left to right of the dividend and multiply the divisor by a certain number or x to cancel out the first terms. At the end, there will either be a remainder or the polynomials will divide completely.



Review Problems

1. Find a polynomial function formula that approximately fits this graph.



2. Factor x^4 - 7x^2 + 9 over R.

3. Use Lagrange interpolation to find a function involving the points (3,3), (4,7), (7,-11), (10,25).

4. Determine the end behavior and the zeroes of -5x^3+5x^2+30x

5. Simplify 2x^5-7x^4-4x^3+18x^2+19x-30 / x^2-5x+6 using first fractional division then long division.

6. Find a function that fits the input-output table below:

Input	Output
1	2
2	5
3	10
4	17
5	28

Solutions

1. Both of the graph's ends go up, so the leading coefficient must be positive and the degree must be even. There are four zeros: -3, -1, 1, and 2, therefore: $a(x+3)(x+1)(x-1)(x-2)$. Now solve for the leading coefficient a using the point (0,6).

$6=a(3)(1)(-1)(-2)$

$6=6(a)$

$a=1$

Final equation: $(x+3)(x+1)(x-1)(x-2)$

2. $x^4 - 7x^2 + 9$

Let $z= x^2$, and substitue z in to get $z^2 - 7z + 9$

$x^4 - 7x^2 + 9 = x^4 - 6x^2 + 9 = (x^2 - x^3)^2 - x^2$ Now this is a difference of squares.

$x^4 - 7x^2 + 9 = (x^2 - 3)^2 - x^2 = (x^2 - 3 - x) (x^2 -3 + x)$

So, $x^4 - 7x^2 + 9 = (x^2 - x - 3) (x^2 + x - 3)$

3. $f(x)=a(x-4)(x-7)(x-10)+b(x-3)(x-7)(x-10)+c(x-3)(x-4)(x-10)+d(x-3)(x-4)(x-7)$

$f(x)=-0.107143(x-4)(x-7)(x-10)+0.388888(x-3)(x-7)(x-10)+0.305555(x-3)(x-4)(x-10)+.198413(x-3)(x-4)(x-7)$

4. End behavior: leading coefficient is -5 and the degree is 3. Since the degree is odd and the leading coefficient is negative, the graph goes up on the left and down on the right.

Zeroes: first factor out -5x, which is a common factor. Then, factor the remaining quadratic.

$-5x^3+5x^2+30x = -5x(x^2-x-6) = -5x(x-3)(x+2)$

Thus, the zeroes are 0, 3 and -2.

5. Starting with fractional division, divide the equation by $(x-3)/(x-3)$

This simplifies the polynomial out to be $2x^4-x^3-7x^2-3x+10 / x-2$

Finally, we use long division and the outcome should be 2x^3+3x^2-x-5.

6.

$28 = 5^2+A(5-1)(5-2)(5-3)(5-4)$

$A(5-1)(5-2)(5-3)(5-4)=3$

$A=\frac{1}{8}$

Function: $F(x) = x^2+\frac{1}{8}(x-1)(x-2)(x-3)(x-4)$