<u>Sequences and Series (Ch. 2)</u>

Investigation 2A: Students must be able to expand $\sum$ notation into equations or derive a sum in $\sum$ notation from a sum. It is necessary to be able to take a sum in the form $\sum_{k=0}^{n} f(k)$ and write it in the form of an algebraic equation, or take an algebraic equation or a sum of numbers such as and convert it to the form $\sum_{k=0}^{n} f(k)$

Students must be able to calculate sums from arithmetic and geometric sequences.

A sequence is an **arithmetic sequence** if its domain is the set of integers $n \geq 0$, and there is a number d, the common difference, for the sequence such that $f(n) = f(n-1) + d$ for all integers $n > 0$. A sequence is a **geometric sequence** if its domain is the set of integers $n \geq 0$, and there is a number $r \neq 0$, called the common ratio, such that $f(n) = r * f(n-1)$ for all integers $n > 0$.

For arithmetic sequences it is best to calculate sums using **Gauss's Method** (example problem: Add up the integers from 1 to 100):

$S = 1 + 2 + 3 + 4 + ... + 100$
$S = 100 + 99 + 98 + 97 + ... + 1$

	S	=	1	+	2	+	3	+	...	+	100
	↓		↓		↓		↓				↓
+	S	=	100	+	99	+	98	+	...	+	1
	2S	=	101	+	101	+	101	+	...	+	101

There are one hundred 101's altogether, so
2S = 100*101
S = 5050

For geometric sequences it is best to calculate sums using **Euclid's Method**, which can be found on page 92 of the textbook.

Investigation 2B (Part 1): Students must be able to develop a list of $\sum$ identities and recognize situations in which you can apply them.

Given a function *f* having a domain that contains the nonnegative integers, the series associated with f is the function defined on the nonnegative integers by:

F(n) = $\sum_{k=0}^{n} f(k) = f(0) + f(1) + f(2) + f(3).... + f(n)$

So the series associated with f is a running total of the outputs of *f*.

Σ Identities:

1. $\sum_{k=0}^{n} cf(k) = c \sum_{k=0}^{n} f(k)$, *where c is any real number*
2. $\sum_{k=0}^{n} (f(k) + g(k)) = \sum_{k=0}^{n} f(k) + \sum_{k=0}^{n} g(k)$
3. $\sum_{k=0}^{n} f(k) = \sum_{k=0}^{m} f(k) + \sum_{k=m+1}^{n} f(k)$, *where* $0 < m < n$
4. $\sum_{k=0}^{n} 1 = n + 1$ *or* $\sum_{k=1}^{n} 1 = n$
5. Think Gauss: $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$
6. Think Euclid: $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$

7.Bernoulli's Formulas: $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$; for $\sum k^3, k^4$, *etc*., *look on page* 114 *in the textbook*

Investigation 2B (Part 2): Find closed form expressions for indefinite sums and for the series associated with a function.

To turn an indefinite sum into a closed form function, use the identities listed above to simplify the sum into parts easily translatable into pieces of a function. Leave the variable *n* as the indefinite. Plug in a value into this function to find any definite sum of the series.

Investigation 2C: Students must be able to find a close-form function for an arithmetic series, as well as find the limit of a geometric series (if one exists). For example, if given an arithmetic series it's important to know how to turn that into a closed-form function. If they have matched an arithmetic series to a recursive function that fits the form: f(n)= $\{\frac{r \text{ if } n=0}{f(n-1)+an\hat{\ }4+bn\hat{\ }3+cn\hat{\ }2+dn+e \text{ if } n>0}$

They should be able to turn that into a closed form function using this formula: f(n)=r+a*(bernoulli formula for n^4)+b*(bernoulli formula for n^3)+c*(bernoulli formula for n^2)+d* $\left(\frac{n(n+1)}{2}\right)$ +e. Students should also be able to find the limit for geometric functions. They should be able to use the $\frac{a}{1-r}$ formula to find the limit of a geometric function when a is the first term and r is the ratio.

Investigation 2D: Students must be able to use combination numbers to express various values. For example, they should be able to express the various values of Pascal's Triangle with combination numbers. An important skill is the ability to write a value in the nth row and kth column as ${}_nC_k$. Furthermore, students must also be able to use the Binomial Theorem to expand expressions of the form $(a + b)^n$. This ties back into the fact that Pascal's Triangle contains the coefficients of a binomial expansion.

For example, row 3 of Pascal's triangle contains the numbers 1, 3, 3, and 1. These are also the coefficients of the expansion of $(a + b)^3$, since $(a + b)^3$
$= a^3 + 3a^2b + 3ab^2 + b^3$. In summary, students should be able to apply combination numbers in other situations.

Vocabulary and Notation:

- arithmetic sequences
- arithmetic series
- Bernoulli's formulas
- closed-form definition
- common difference
- common ratio
- definite sum
- difference table
- Euclid's method
- figurative number
- Gauss's method
- geometric sequence
- geometric series
- identity
- indefinite sum
- index
- limit
- Pascal's Triangle
- recursive definition
- repeating decimal
- sequence
- series associated with *f*
- slope
- term
- summation
- (nCk) (the *n*th row, *k*th column entry of Pascal's triangle)

- $\sum_{k=0}^{n} f(k)$

PROBLEMS

From Investigation 2A:
1. Calculate the sum of all even numbers from 6 to 98, inclusive.
2. Calculate the sum of 1+3+9+27+...+3^9+3^10
3. Derive the Σ notation of 1+r+r^2+r^3+...+r^n-1+r^n

Solutions to problems from Investigation 2A:
1. 2444
2. 88573
3. $\sum_{k=0}^{n} r^k$

1. From Investigation 2D: What is the coefficient of the a^3b^{10} term in the expansion of $(a + b)^{13}$?

1. We can see that the coefficient of the a^3b^{10} term is $_{13}C_3$, since for each of the 13 binomials, there is either a choice of a or b. We must choose 3 a's out of the 13 possibilities. We can simplify $_{13}C_3$ to get that the coefficient of a^3b^{10} is (13!)/(10!*3!), which equals 286.

Investigation 2C:
2a) Provide a closed form function formula for this recursive function:

$$f(x)=\begin{cases} 5 \text{ if } n=0 \\ f(n-1)+5n^2+4n+6 \text{ if } n>0 \end{cases}$$

2b) For what ratio(r-values) will there be no limit to a geometric function:

3. From Investigation 2B Part 1: Evaluate $\sum_{k=0}^{7}(4k+7)$ using the identities presented in given section.

Using identity 2: $\sum_{k=0}^{7} 4k + \sum_{k=0}^{7} 7$

Using identity 1: $4\sum_{k=0}^{7} k + 7\sum_{k=0}^{7} 1$

Using identity 5 and 4: $4 \cdot \frac{7(7+1)}{2} + 7 \cdot (7+1)$

Alegbra: Answer=168

4a) Find the closed form for the indefinite sum: $\sum_{k=1}^{n}(5k+3)$

4b) Use the closed form to find the definite sum of $\sum_{k=1}^{32}(5k+3)$

Solution

a) $\sum_{k=1}^{n}(5k+3)$

$= \sum_{k=1}^{n} 5k + \sum_{k=1}^{n} 3$

$= 5\sum_{k=1}^{n} k + \sum_{k=1}^{n} 3$

$= 5(\frac{n(n+1)}{2}) + 3n$

b) $\sum_{k=1}^{32}(5k+3) = 5(\frac{n(n+1)}{2}) + 3n$

$= 5(\frac{32(32+1)}{2}) + 3(32)$

=10656

2a) $11+5*\frac{n(n+1)*(2n+1)}{6} + 4*\frac{n(n+1)}{2}$

2b) If r is ≤ -1