Monica, Maeve, John, Luke, Lewis & Viraj

TRIGONOMETRY AND GEOMETRY

Essential Knowledge:

- Understand and apply the unit circle definitions of the trig. functions and use the unit circle to find function values using special triangles and w/ a calculator (Monica)
- Prove and apply the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$ (Monica)
- Find trig. function values when given other values using quadrant relationships, the Pythagorean identity, and other identities
- Solve trig. equations by hand, using inverses on the calculator, and graphically on the calculator (John)
- Graph the functions sin x, cos x and tan x and identify the periods of the graphs (Luke)
- Prove and apply the angle sum identities (p. 346) (Viraj)
- Prove and apply the Law of Sines and Law of Cosines
- Solve triangles when given SSS, SAS, ASA, or AAS and in the potentially ambiguous case SSA (Lewis)
- Find areas of triangles using (1/2)a*b*sinC and using Heron's Formula: $\sqrt{s(s-a)(s-b)(s-c)}$

Descriptions of Essential Knowledge:

- The unit circle and its significance:
  - Let θ be an angle centered at the origin and measured from the positive x-axis. The terminal side of θ intersects the graph of $x^2 + y^2 = 1$ (the unit circle) in exactly one point.
  - Using θ and the known length of the radius, which is 1 in the unit circle, you can use cosine and sine to find the coordinates of a point on the unit circle. Because the lengths of the two legs of the right triangle (that has the radius as its hypotenuse and part of the x axis as one of its legs) are just the x and y coordinates of the point on the unit circle, you will ultimately discover that these the x and y coordinates are cosine θ and sine θ, respectively. To figure out whether the coordinates of the point on the unit circle are positive or negative, you can use what you know about the signs in each of the four quadrants. (For practice and to make sure you understand how this works, pick a random angle for θ and try to find the coordinates of the point on the unit circle using the information you have).
  - In other words (under the conditions of the first bullet above):
    The cosine of angle θ is the x-coordinate of this intersection.
    The sine of angle θ is the y-coordinate of this intersection.
    The tangent of angle θ is sinθ/cosθ, whenever $\cos\theta \neq 0$.
- Theorem 4.2: The Pythagorean Identity (pg. 322 in textbook):

- $\sin^2\theta + \cos^2\theta = 1$
- Theorem 4.1 (pg. 317 in textbook):
  - cos (x+360*n) = cos (x)
  - sin (x+360*n) = sin (x)
- Trig Identities :
  - sin(a) = opposite/hypotenuse
  - cos(a)= adjacent/hypotenuse
  - tan(a) = opposite/adjacent
  - csc(a) = 1/sin(a) or hypotenuse/opposite
  - sec(a) = 1/cos(a) or hypotenuse/adjacent
  - cot(a) = 1/tan(a) or adjacent/opposite

- Angle Sum Identities:

There are two main angle sum identities:
- $sin(\alpha + \beta) = sin(\alpha)cos(\beta) + cos(\alpha)sin(\beta)$
- $cos(\alpha + \beta) = cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta)$

Others identities can be derived from these two. An example of this would be the angle difference identities which are:
- $sin(\alpha - \beta) = sin(\alpha)cos(\beta) - cos(\alpha)sin(\beta)$
- $cos(\alpha - \beta) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$

There are two special cases as well that occurs when the angle is the same for the angle sum identity - cos(2α) and sin(2α)
- $cos(2x) = cos^2(x) - sin^2(x)$
- $sin(2x) = 2sin(x)cos(x)$

The full proof of the angle sum identities can be found on pg. 347 of the textbook.

- Completing Triangles:

<u>AAS</u>: Label a triangle the standard way. You know side length a, ∠A, ∠B.

1. Find ∠C by using ∠C=180-∠B-∠A.

2. Apply Law of Sines with C/A.
(sinC/c)=(sinA/a)

3. Cross-multiply.
a*sinC=c*sinA

4. Divide by sinA on both sides.

c=(a*sinC/sinA)

5. Repeat steps 1-4 for side b.

SAS: Label triangle ABC in a standard way. Given: a, b, ∠C.

1. Use law of cosines:

$a^2 + b^2 - 2ab\cos C = c^2$
$c = \sqrt{(a^2 + b^2 - 2ab\cos C)}$

$\cos A = (c^2+b^2-a^2)/2ab$
$\angle A = \cos^{-1}((c^2+b^2-a^2)/2cb)$

∠B = 180-(∠A+∠B)

ASA: Label triangle ABC in a standard way. Given: ∠A, b, ∠C.

1. ∠B = 180 - (∠A+∠B)

2. (SinA/a) = (SinB/b)
   a*SinB = b*SinA
   a = (bSinA)/SinB

3. Solve side c in the same way as side a

SSS: Label triangle ABC in a standard way. Given: a, b, c.

$\angle A = \cos^{-1}((c^2+b^2-a^2)/2cb)$
$\angle B = \cos^{-1}((c^2+a^2-b^2)/2ca)$
$\angle C = \cos^{-1}((c^2+a^2-b^2)/2ab)$ or $180 - (\angle A + \angle B)$

SSA: Label triangle ABC in a standard way. Given: a, b, ∠C(not opposite to side C).

$c = \sqrt{(a^2 + b^2 - 2abcosC)}$

∠A = $\sin^{-1}((a*SinC)/c)$

but check the relationship of a b and c

if($a^2 + b^2 < c^2$)

then triangle is acute

if($a^2 + b^2 > c^2$)

then triangle is obtuse

if($a^2 + b^2 = c^2$)

then it is right

find the correct angle for the sin value

then ∠B = 180 - (∠A + ∠C)

- Solving Trig Equations:

Method 1:

1. Replace all sin(θ)'s or cos(θ)'s with a variable. (but first make sure the equation is with one trig function only)

ex: $\sin^2(\theta) + 2\times\sin(\theta) + 1 = 0$

call sin(θ) = x

$x^2+2x+1=0$

2. Solve the new equation as you would a normal equation.

ex. $x^2+2x+1=0$

$(x+1)^2=0$

x = -1

3. Replace x back with sin(θ) or cos(θ).

ex. x = -1

sin(θ)= -1

4. Use the $\sin^{-1}(x)$ function on your calculator to find θ.

ex. sin(θ)= -1

$\sin^{-1}(-1)= -90^o$ or -π/2.

5. Add the period of the original trig equation times n, where n is an integer, to your solution.

ex. $-90^o + 360^o n$ or -π/2+2πn, where n is an integer. (sin(θ) has a period of 360^o or 2π)

Method 2:

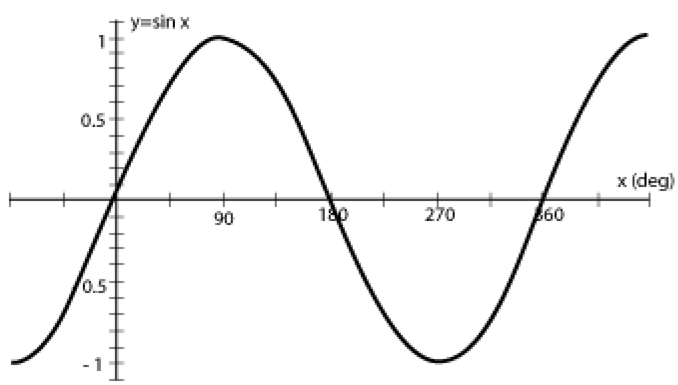
1. Move everything in the trig equation involving a trig function to one side of the equals sign.
ex. instead of $\sin^2(\theta) = 2\sin(\theta) + 1$, you should have $\sin^2(\theta) + 2\sin(\theta) + 1 = 0$

2. Put the left side of the trig equation into the y= part of your calculator.
ex. $\sin^2(\theta) + 2\times\sin(\theta) + 1 = 0$
(in calculator) $y_1=\sin^2(\theta) + 2\times\sin(\theta) + 1$

3. Put the right side into the next line.
ex. (in calculator) $y_2=0$
4. Using [2nd][CALC] and then "5: intersect", find the intersections between the two graphs.

- Graph the functions sin x, cos x and tan x and identify their periods.

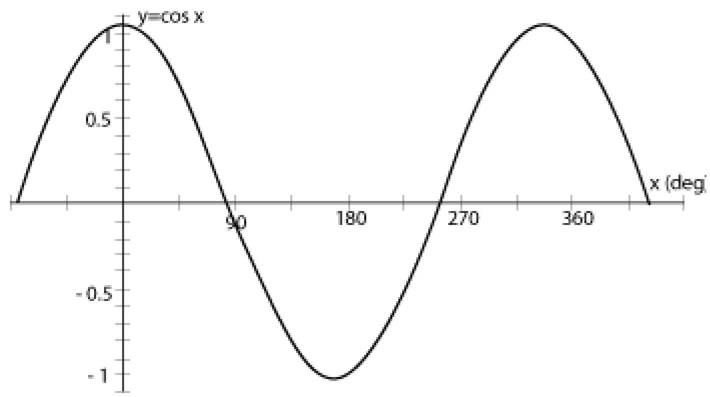
Sin(x)



The graph of Sin(x) can be created by taking the vertical, opposite side of the unit circle triangle and dividing it by the hypotenuse. Since the hypotenuse is always 1 in the unit circle, Sin(x) can be graphed as the changing length of the vertical side of the unit circle triangle caused by different degrees of the triangle.

The period of a graph is the distance for the graph to complete one full cycle. In Sin(x), the period is 360 degrees.

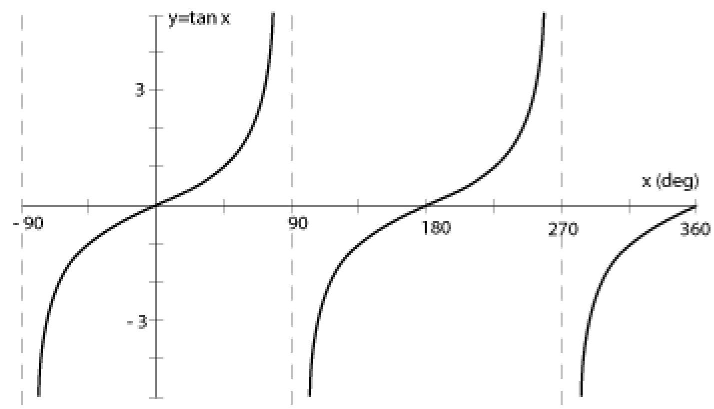
Cos(x)



The graph of Cos(x) can be created by taking the horizontal, adjacent side of the unit circle triangle and dividing it by the hypotenuse. Cos(x) can be graphed as the changing length of the horizontal side of the unit circle triangle caused by different degrees of the triangle.

The period of Cos(x) is 360 degrees, like Sin(x).

Tan(x)



The graph of Tan(x) can be created by taking the vertical, opposite side of the unit circle triangle and dividing it by the horizontal, adjacent side. Tan(x) can be graphed as the changing ratio of the vertical to the horizontal sides caused by different degrees of the triangle. Asymptotes of Tan(x) can be found with 90+180n, where n is an integer.

Unlike Sin(x) and Cos(x), the period of Tan(x) is 180 degrees.

Graph the functions sin x, cos x and tan x and identify their periods. The graphs of these trigonometric functions can be created through knowing and using the definitions of trigonometric functions in terms of ratios between different sides. Trigonometric graphs Sin(x) and Cos(x) have periods of 360 degrees, which means they repeat after every 360 degrees. Tan(x) has a period of 180 degrees, and has vertical asymptotes every half rotation starting from 90 degrees, due to the division by 0.

<u>Problems:</u>

1. Prove the Pythagorean Identity: $\sin^2\theta + \cos^2\theta = 1$
2. Prove the Law of sines and Law of Cosines
3. Solve for $\sin(105°)$ without using a calculator. *Hint - use angle sum identities.*
4. Find all solutions for θ in $\cos^2(2\theta) = \sin^2(2\theta) + \sin(2\theta)$.
5. Express each value as a sine or cosine of an acute angle:
    a. $\sin 400°$
    b. $\cos 400°$
    c. $\cos 300°$
    d. $\sin(-100°)$

Express each value as a sine or cosine of θ:

    e. $\sin(180° - \theta)$
    f. $\cos(180° - \theta)$
    g. $\sin(180° + \theta)$
    h. $\cos(180° + \theta)$
    i. $\sin(270° - \theta)$
    j. $\cos(270° - \theta)$
    k. $\sin(270° + \theta)$
    l. $\cos(270° + \theta)$

Suppose an angle θ is unknown, but it is between 300° and 350°. Determine whether each expression is positive or negative:

    m. $\cos\theta$
    n. $\sin\theta$
    o. $\tan\theta$
    p. $\cos^2\theta + \sin^2\theta$
    q. $\tan^3\theta$

6. If Tan(x) and Sin(x) were put on the same graph, name all the angle values that would contain no values for Tan(x) and the lowest values for Sin(x).
7.
8.

<u>Answers:</u>

Solution to Problem 1:

$a^2 + b^2 = c^2$

Divide each side by c^2.
$a^2/c^2 + b^2/c^2 = 1$
$\cos\theta = a/c$ and $\sin\theta = b/c$, so $(\cos\theta)^2 + (\sin\theta)^2 = 1$

Solution to Problem 2:
Law of Sines--Set up triangle ABC with sides abc opposite corresponding angles. Then draw in a height from B to side b. That means that

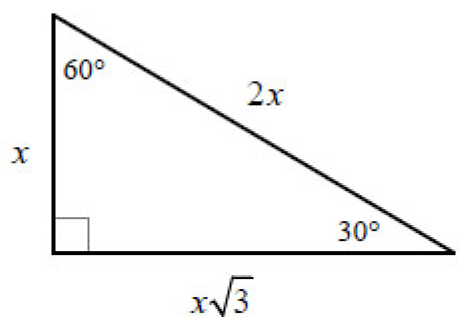
SinA=h/c so h=cSinA
SinC=h/a so h=aSinC
cSinA=asinC
a/SinA=c/SinC

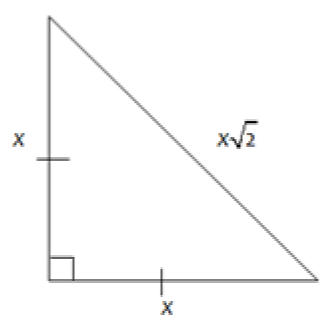
Same reasoning to prove a/SinA=b/SinB=c/SinC

Law of Cosines--Draw the same triangle with the same height B to b. Label from C to where the height interests AC "aCosC" and "b-aCosC". Then you can use pythag. theorem.

c^2= (aSinC)^2 + (b-aCosC)^2
which simplifies to…
=a^2 + b^2 + 2abCosC

Solution to Problem 3:
Split the angle 105° into the special angles - 60° and 45°. Now you will have sin(60° + 45°). This will expand into sin(60)cos(45) + sin(45)cos(60) using the angle addition identity. These two angles make special triangles [60° for the 30-60-90 triangle and 45° for the 45-45-90 triangle] and can be solved without using a calculator. By using the known ratios of the special triangles, solve for sin(60°),cos(60) and sin(45°), cos(45). Use the basic trigonometric identities (ex. sin(a) = opposite/hypotenuse). Sin(60°) will turn out to be $\sqrt{3}/2$ and sin(45°) will be $\sqrt{2}/2$. Cos(60°) will turn out to be 1/2 and cos(45°) will be $\sqrt{2}/2$. Plug these into the original equation, <u>the answer will be $(\sqrt{6}+\sqrt{2})/4$.</u>





Solution to Problem 4:

1. Express the equation in terms of one trig function using the Pythagorean Identity.
$\cos^2(\theta) = \sin^2(2\theta) + \sin(2\theta)$
$1 - \sin^2(2\theta) = \sin^2(2\theta) + \sin(2\theta)$

2. Replace $\sin(2\theta)$ with x.
$1 - x^2 = x^2 + x$

3. Solve the equation for x.
$2x^2 + x - 1 = 0$
$(2x-1)(x+1) = 0$
$x=½, -1$

4. Using $\sin^{-1}(x)$ on your calculator, find the value of 2θ.
$\sin^{-1}(½) = 2\theta = 30^o$ or $\pi/6$

5. Add "$+360^o n$" or "$+2\pi n$" (depending on whether you used radians or degrees) to your answer.
$2\theta = 30^o + 360^o n$ or $\pi/6 + 2\pi n$

6. Divide by 2 to find θ.
$\theta = 15^o + 180^o n$ or $\pi/12 + \pi n$.

Solution to Problem 5:
(note there may be more than 1 answer to a problem)
a. sin40 or cos50
b. cos40 or sin50
c. cos60 or sin30
d. -sin80 or -sin20
e. sinθ
f. -cosθ
g.-sinθ
h.-cosθ
i.-cosθ
j.-sinθ
k.-cosθ
l. sinθ
m. positive
n. negative
o. negative
p. positive

q. negative

Solution to Problem 6:
This question can be simplified to mean - “Determine which Tan(x) asymptotes would lie on minimum Sin(x) values. Since Sin(x) is the lowest at 270+360n degrees since it has the lowest y value, we can consider it as a possible solution. Since Tan(x)’s asymptotes also fall on 90+180n degrees (in both equations, n is an integer), the solution to the problem is 270+360n degrees because this includes all the degree values that create both the lowest Sin(x) values and the Tan(x) asymptotes.