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Solids and Volume

Key Concepts:

- Cavalieri's principle

Cavalieri's principle states that if two solids of the same height are cut by a plane so that the cross sections produced have the same area, and then are cut again by a plane parallel to the first and the resulting cross sections still have the same area, then they have the same volume. For example, if you have a stack of coins (as shown below), no matter what position the stack is in, common sense shows the stack will have the same volume. To show this using Cavalieri's principle, you can cut the stack along a plane. The area of the cross section will be the same, because it will be a cross section same kind of coin. Therefore, every plane's cross section will have the same area because all the coins have the same area.

- Proof of Square Pyramid Formula ($\frac{1}{3} s^2 h$)

The square pyramid formula is proven theoretically using an imaginary cube with side length *s*. The cube can be dissected along its diagonals to create 6 identical square pyramids, each with ⅙ the volume of the cube. Because the volume of the cube is s^3, the volume of each pyramid must be $\frac{1}{6} s^3$. Each pyramid has height $h = \frac{1}{2} s$ and area $B = s^2$. The volume of each pyramid can then be rewritten:

$V = \frac{1}{6} s^3$

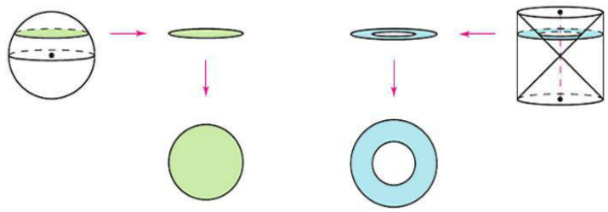
$V = \frac{1}{3} * s^2 * \frac{1}{2} s$

$V = \frac{1}{3} B * h$

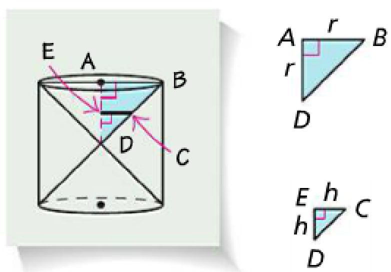
- Proof of the Volume of a Sphere ($4/3\ \pi\ r^3$)

The volume of a sphere is proven by first proving that the circle and the ring formed at the intersection of the plane at height h with a sphere and with a cylinder with a double cone removed have the same area and then finding the volume of the cylinder with the double cone removed, which is the same as the volume of the

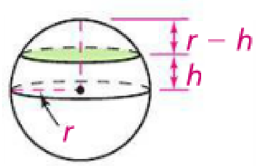
sphere. You have a sphere and a cylinder with a double cone removed from it. The sphere has radius r, and the cross section of the sphere is at a distance h from the center of the sphere. The cylinder with the double cone removed from it has radius r and height 2r. The cross section of the double cone is also at height h from the center of the cylinder.



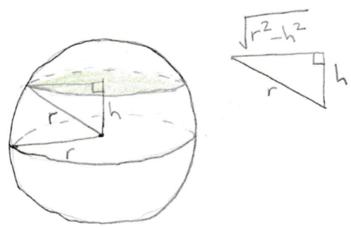
Point A is the center of one of the bases of the cylinder; point B is on the outer edge of the same base of the cylinder, and point D is the point where the two removed cones meet. Point E is anywhere along the center line of one of the cones, and the distance from D to E is h. Point C is along the edge of the same cone, and EC is parallel to AB. Triangle ABD is an isosceles right triangle because the cones both have base radius r and height r. Since triangle ECD is similar to triangle ABD, then triangle ECD is also an isosceles right triangle. Thus EC is equal to h. Then the area of the ring formed by cutting along the plane going through EC is $\pi r^2 - \pi h^2$, which equals $\pi(r^2 - h^2)$.



Now it is necessary to find the area of the intersection of the plane at height h with the sphere.



Draw a line from the center of the sphere to a point on the edge of the sphere where the plane at height h intersects the sphere. The length of this line is r since it is a radius of the sphere, and this line helps create a right triangle, as shown below. One of the legs of this triangle has length h, and the hypotenuse has a length of r. Using the Pythagorean theorem, the length of the other leg of this triangle is the square root of $(r^2 - h^2)$.



Therefore, the circle formed by the intersection of the plane at height h with the sphere has an area of π(sqrt.(r^2 - h^2))^2. This simplifies to π(r^2 - h^2), which is the same as the area of the ring that is formed by the intersection of the plane at height h with the cylinder with a double cone removed from it. So no matter what h equals, the areas of the circle and of the ring will always be the same. Thus the volume of the cylinder with the double cone removed is equal to the volume of the sphere, for both these shapes have a height of 2r, and the circle and ring formed at the intersection of the sphere or cylinder and the plane at height h have equal areas.

The cylinder has a volume of $2r*\pi r^2$, or $2\pi r^3$. The volume of one of the cones is (⅓)*πr^2*r, or (πr^3)/3. Thus the area of the cylinder with the double cone removed is $2\pi r^3$-(⅔)*πr^3. This simplifies to (4/3)*πr^3. Since the volume of the cylinder with the the double cone removed is equal to the volume of the sphere, the volume of a sphere is (4/3)*πr^3.

Essential Ideas:

Volume Formulas

Sphere……………………..$V = 4/3\ \pi\ r^3$

Any Prism or Cylinder...….V = Bh (area of base x height)

Any Cone or Pyramid…......V = 1/3Bh (one-third area of base x height)

Cavalieri's Principle

(see Key Concepts section)

Review Problems:

1) A candle holder is a right square pyramid made of glass with side length x = 3.5 inches and height y = 4 inches. It has just enough space hollowed out inside to hold a candle in the shape of a right circular cylinder (diameter = 1.4 inches, height = 3 inches). Glass weighs 1.6 oz per cubic inch. How many pounds does the candle holder weigh without a candle inside?

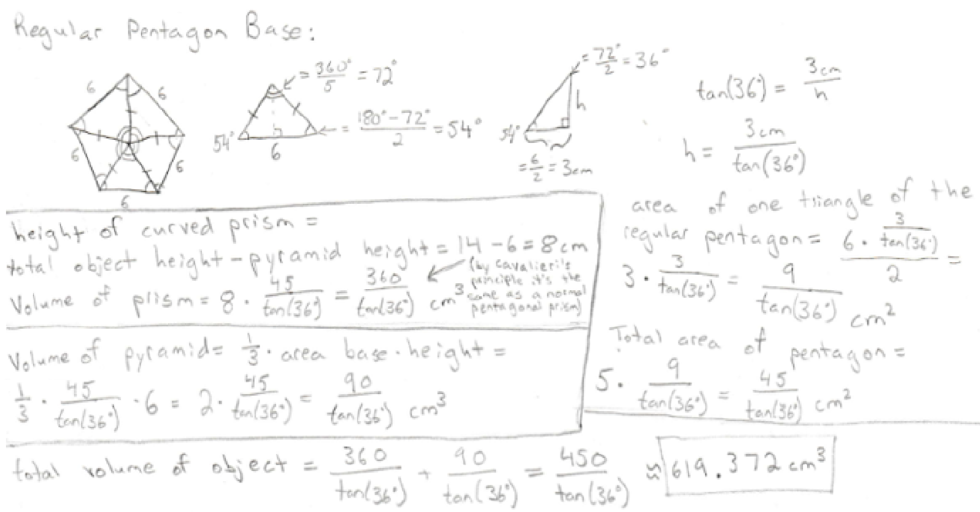
2) This object consists of a pyramid on top of a curved prism. The bases of both parts of the object are regular pentagons with a side length of 6 cm. The height of the pyramid part of this object is also 6 cm, and the total height of this object is 14 cm. What is the total volume of the object?

3) There are 2 similar cylinders. One has a volume of 84 and the other has a volume of 21. The base area of the larger cylinder is 16π. What is the height of the smaller cylinder?

Solutions to Review Problems:

1) The candle holder weighs 1.172 pounds. The volume of the un-hollowed-out pyramid is ⅓ * $(3.5)^2$ * 4 = 16.333 inches^3. Subtract the volume of the candle (π * $(1.4/2)^2$ * 3 = 4.618 inches^3) to get 11.715 inches^3, the volume of the glass. Multiply by 1.6 to get 18.744 oz of glass, which is 1.172 pounds.

2) about 619.372 cm^3



3) The height of the smaller cylinder is 1.053 units. The larger cylinder's height will be 84/16π or 1.671 units, because of the volume formula for a cylinder. Take the cube root of the ratio of the volumes so it can be compared to the heights. Therefore, the proportion is $\sqrt[3]{84}/\sqrt[3]{21} = 1.671/x$. If you cross multiply, you get $\sqrt[3]{84}x = 4.61$. Then, if you divide by $\sqrt[3]{84}$, you get 1.053 units as the height of the smaller cylinder.