Determinants

For every square matrix \( A \), there is a number called the determinant of the matrix, denoted as \( \det(A) \) or \( |A| \). Sometimes the bars are written just around the numbers of the matrix, such as
\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix}
\]
which means the same as \( \det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) \).

Finding values of determinants

Here are methods for finding the determinants for matrices of various sizes. For some worked-out examples, see pp. 458–459 in the purple Advanced Math book.

- **1-by-1 matrix**: The determinant is just the single number of the matrix.

- **2-by-2 matrix**: \[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.
\]

- **3-by-3 matrix**: Found by combining three different 2-by-2 determinants in the following way:
\[
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}.
\]

Note the pattern: Each entry of the top row is multiplied by the 2-by-2 determinant that you get by omitting its entire row and entire column. Also note the alternating + and – signs.

- **n-by-n matrix**: Follows a similar pattern. Each of the \( n \) numbers in the top row is multiplied by an \( (n-1) \) by \( (n-1) \) determinant (found by omitting the entire row and entire column). Then combine by alternately adding and subtracting.

- **On the calculator**: After entering a matrix in the calculator in the usual way, you can find its determinant by applying the \texttt{det} function to it (found under the Matrix key, Math submenu).

Uses of determinants

Determinant calculations are important because they appear in connection with several topics we’ve studied in this course. Here are the uses of determinants which will be described in the following pages.

- Finding out whether a given matrix has an inverse, and for 2-by-2’s, finding the inverse.
- Finding out whether a linear system has a unique solution.
- Finding out some of the geometric characteristics of a linear transformation.
- Finding areas and volumes of certain shapes formed by vectors.
- Stating the definition for the cross-product of vectors.

Looking for more material on these topics beyond this packet? Generally the purple Advanced Mathematics book is best. See sections 12-7 (calculating determinants), 12-8 (area and volume), 12-9 (vector cross product), and 14-6 (transformation matrices).

(Yet another application of determinants is a method of solving linear systems called Cramer’s Rule, which some of you may have studied in Algebra 2. See Advanced Math section 12-8 if you want to learn or relearn this method. However, it won’t be required in this course because the row operation (rref) is a much more effective method for solving systems.)
Inverse matrices

You can use determinants to find out whether a given matrix $A$ has an inverse matrix:

- $A^{-1}$ exists $\iff$ $\det(A) \neq 0$.
- If $\det(A) \neq 0$, then $A^{-1}$ exists, and it’s said that $A$ is invertible.
- If $\det(A) = 0$, then $A^{-1}$ doesn’t exist, and it’s said that $A$ is singular.

Additionally, for 2-by-2 matrices, the determinant is helpful in finding the inverse matrix. In a previous homework assignment, you derived this formula for 2-by-2 inverses:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note that the denominator in this formula is the determinant.

Solutions to linear systems

Recall this method we studied for solving linear systems using an inverse matrix. Any system of $n$ linear equations in $n$ variables can be combined into a single matrix equation of the form $AX = B$, where $A$ is a matrix of coefficients, $X$ is a one-column matrix of variables, and $B$ is a one-column matrix of constants. Then the system can be solved using these steps:

$AX = B \implies A^{-1}AX = A^{-1}B \implies IX = A^{-1}B \implies X = A^{-1}B$.

However, the above method does not work for all linear systems. It depends on the existence of $A^{-1}$. Not all matrices have inverses. But now, you can use a determinant to find out whether or not the inverse exists, and whether or not the system has a unique solution.

- If $\det(A) \neq 0$, then $A^{-1}$ exists, and any linear system that can be written in matrix form $AX = B$ will have a unique solution given by $X = A^{-1}B$.
- If $\det(A) = 0$, then the linear system will not have a unique solution. (It might have no solutions or infinitely many solutions, but neither $\det(A)$ nor $A^{-1}$ can help you find out which; you’ll have to use the row operations (rref) approach instead.)

**Specific example:** The linear system $5x + 4y = -2$, $2x + 3y = -5$ can be written as

$$\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

The solution of the system above is given by $X = A^{-1}B$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$
**Transformation matrices**

Suppose that $T$ is the 2-by-2 matrix of a linear transformation. The value of the determinant of $T$ tells several important things about the transformation.

- The absolute value of the determinant tells the area scaling factor of the transformation:
  
  If $R$ is a region of the plane and $R'$ is its image, then area of $R'$ = $| \det(T) | \cdot $ area of $R$.
  
  (For example, if $\det(T) = 12$, every region is mapped to a region that is 12 times as large.)

- If $\det(T) > 0$, then the transformation is *orientation preserving*. That is, $R$ and $R'$ will have the same orientation. (For example, all rotations have a positive determinant.)

- If $\det(T) < 0$, then the transformation is *orientation reversing*. That is, $R$ and $R'$ will have opposite orientations. (For example, all reflections have a negative determinant.)

- If $\det(T) = 0$, then the transformation maps the plane to either a line or a point.
  
  (For example, all projections have a zero determinant.)

**Finding areas using 2-by-2 determinants**

**Parallelograms**: Suppose the sides of the parallelogram are described by vector $u$ (one pair of parallel sides) and vector $v$ (the other pair of parallel sides). Suppose $u = <u_1, u_2>$ and $v = <v_1, v_2>$. Then the parallelogram is the absolute value of $\det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$, which is $|u_1v_2 - u_2v_1|$.

**Triangles**: Since any triangle is half of a parallelogram (formed by attaching a congruent copy of the triangle, rotated 180°, to one of the triangle’s sides), the triangle’s area is half of the parallelogram’s area. So, if vectors $u = <u_1, u_2>$ and $v = <v_1, v_2>$ form two of the sides of a triangle from a common starting point, then the area of the triangle is $\frac{1}{2}$ of the absolute value of $\det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$, which equals $\frac{1}{2} |u_1v_2 - u_2v_1|$.

**Definition of the cross-product of vectors**

As we discussed in our vector unit, an easy way to remember the formula for the cross-product of vectors $u = <a, b, c>$ and $v = <d, e, f>$ is to evaluate this 3-by-3 determinant:

$$ u \times v = \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} $$

where $i$, $j$, and $k$ stand for unit vectors $i = <1,0,0>$, $j = <0,1,0>$, $k = <0,0,1>$. 
**Problem set**

1. Without using your calculator, find the determinant of each matrix below, and tell whether or not the matrix has an inverse.
   
   a. \[
   \begin{bmatrix}
   3 & 4 \\
   -2 & 5 \\
   \end{bmatrix}
   \]
   
   b. \[
   \begin{bmatrix}
   -6 & 3 \\
   4 & -2 \\
   \end{bmatrix}
   \]
   
   c. \[
   \begin{bmatrix}
   2 & 1 & 3 \\
   -6 & 3 & 4 \\
   4 & -2 & 5 \\
   \end{bmatrix}
   \]

2. Calculate the inverse of \[
   \begin{bmatrix}
   3 & 4 \\
   -2 & 5 \\
   \end{bmatrix}
   \] using the formula on page 2.

3. Which of the following linear systems have a unique solution? Decide using determinants. For those that do have unique solutions, find the solutions using the \(A^{-1}B\) method. (You may use your calculator. It’s easiest to enter matrices A and B, then ask the calculator to do \([A]^{-1}[B]\) as a single calculation.)
   
   a. \[3x + 4y = 29\]
   \[-2x + 5y = -4\]
   
   b. \[2x + y + 3z = 15\]
   \[-6x + 3y + 4z = 24\]
   
   c. \[4x + 2y = 12\]
   \[2x + y = 6\]
   
   d. \[4x + 2y = 13\]
   \[2x + y = 6\]
   \[4x - 2y + 5z = 33\]

4. Identify these commonly-seen transformation matrices, and find the determinant of each.
   
   a. \[
   \begin{bmatrix}
   0 & 1 \\
   1 & 0 \\
   \end{bmatrix}
   \]
   
   b. \[
   \begin{bmatrix}
   h & 0 \\
   0 & v \\
   \end{bmatrix}
   \] (assuming \(h\) and \(v\) are positive numbers)
   
   c. \[
   \begin{bmatrix}
   \cos \theta & -\sin \theta \\
   \sin \theta & \cos \theta \\
   \end{bmatrix}
   \]
   
   d. \[
   \begin{bmatrix}
   0 & 0 \\
   0 & 1 \\
   \end{bmatrix}
   \]
   
   Based on the determinant values, tell whether each transformation is *orientation preserving*, *orientation reversing*, or neither. (Make sure that the answer fits with your knowledge of what type of transformation it is.)

5. From your work in problem 4, and your knowledge of other matrices for the same type of transformation, complete a *conjecture* (not a proof) about each of the following:
   
   a. The determinant of a rotation around the origin is always …
   
   b. The determinant of a reflection is always …
   
   c. The determinant of a dilation is always …
   
   d. The determinant of a projection is always …

6. The transformation matrix \[
   \begin{bmatrix}
   \frac{1}{2} & \frac{\sqrt{3}}{2} \\
   \frac{\sqrt{3}}{2} & -\frac{1}{2} \\
   \end{bmatrix}
   \]
   describes either a reflection, a rotation, or a projection. Which type of transformation is it? (Hint: use your conjecture from problem 5.)
7. Answer the following questions about the transformation \( x' = 3x + 4y, \ y' = -2x + 5y \).
   a. What is the area scaling factor for this transformation?
   b. Suppose \( R \) is the rectangle with vertices (1, 2), (3, 2), (1, 5), and (3, 5).
      Apply the above transformation to \( R \). What shape does the image region have?
      What are the coordinates of the vertices of the image region?
      What is the area of the image region?
   c. Suppose this transformation is applied to the unit circle. What is the area of the image region? Why is it harder to specifically identify the shape of the image region?

8. Consider the parallelogram with vertices \( (2, 4), (3, -1), (6, 10), \) and \( (7, 5) \).
   a. Let \( u \) and \( v \) be the vectors representing the sides of the parallelogram starting from \( (2, 4) \).
      Find the components of \( u \) and \( v \).
   b. Find the area of the parallelogram using a determinant.

9. Consider the triangle with vertices \( (-2, 5), (4, -7), \) and \( (6, 1) \).
   a. Find the area of this triangle without using a determinant.
   b. Find the area of this triangle using a determinant. 
      \( \text{Hint: Start by choose two sides of the triangle as } u \text{ and } v, \text{ then finding the component forms of } u \text{ and } v. \)
   c. Which of these two calculations was easier? Which would you use in the future?