Vectors and geometry

Objective: Use vectors to represent geometric shapes and prove geometry theorems.

Parallel vectors

Definition: We say that two vectors are parallel if they lie on lines that have the same slope.¹

1. Consider the vectors \( \mathbf{v} = \langle -2, 5 \rangle \) and \( \mathbf{w} = \langle 10, -25 \rangle \).
   a. Using the definition, prove that vectors \( \mathbf{v} \) and \( \mathbf{w} \) are parallel.
   b. Express \( \mathbf{w} \) as a scalar multiple of \( \mathbf{v} \). That is, express \( \mathbf{w} \) as a number multiplied by \( \mathbf{v} \).

2. Consider the vectors \( \mathbf{v} \) and \( k\mathbf{v} \) (where \( k \) stands for a non-zero real number).
   a. Prove that \( \mathbf{v} \) and \( k\mathbf{v} \) are parallel.
      
      Hints: Begin by writing \( \mathbf{v} = \langle a, b \rangle \). Find \( k\mathbf{v} \) in component form. Then apply the definition of parallel for vectors.
   b. When you have vectors \( \mathbf{v} \) and \( k\mathbf{v} \), how are the lengths of their arrows related?

Going forward, we will take the results of problem 3 to be established tools. For example, if the goal is to prove that \( \mathbf{v} \) and \( \mathbf{w} \) are parallel, it will suffice to show that \( \mathbf{w} \) is a scalar multiple of \( \mathbf{v} \) (for example, that \( \mathbf{w} = \frac{1}{2} \mathbf{v} \)). Slope calculations will no longer be needed.

¹ A necessary special case: the vector \( \langle 0, 0 \rangle \) does not determine a line so we can’t use this definition as stated. Instead, we just define that \( \langle 0, 0 \rangle \) is considered to be parallel to every vector.
Proving a theorem about triangle midsegments

Consider \( \triangle ABC \) with D and E as midpoints of sides, as shown. There is a two-part theorem about how the midsegment DE relates to the triangle. Today we are going to re-prove this theorem using vectors. Fill in the two goals of this proof.

Proof goal 1: Show that

Proof goal 2: Show that

4. This problem will lead you through the proof. Let \( \vec{v} = \overrightarrow{AB} \) and \( \vec{w} = \overrightarrow{AC} \).
   a. Identify each of these vectors in terms of \( \vec{v} \) and \( \vec{w} \):
      \[
      \overrightarrow{BC} = \\
      \overrightarrow{AD} = \\
      \overrightarrow{AE} = \\
      \overrightarrow{DE} = \\
      
    \]
   b. Prove that \( \overrightarrow{DE} \) is a scalar multiple of \( \overrightarrow{BC} \).
      That is, show that \( \overrightarrow{DE} \) equals a number times \( \overrightarrow{BC} \).

    c. Explain how the two parts of the proof goal are now met.
Postscript: why vector proofs

Why do we need vector proofs of geometry theorems? Once you’ve learned how to use vectors, vector proofs are often both easy to carry out and easy to remember. That might not be true for the ordinary geometry proofs for the theorems. (As a test of this, you proved the triangle midsegment theorem 2½ years ago Math 2. Do you remember the proof? Could you do the proof right now? But the vector proof is one that you could realistically remember years from now.)

Homework

5. Show that three of these vectors are parallel to each other, but that the fourth vector isn’t parallel to the others: \(<6, -4>, <9, -6>, <12, -9>, <15, 10>\).

6. This problem will lead you through a vector proof that the diagonals of a parallelogram bisect each other. For this problem, you must take this fact as not already proved.

Here is the central strategy of this proof: We are going to determine the midpoints of the two diagonals separately, then argue that they have to be the same point.

Start with this notation: call the parallelogram ABCD, with \(\mathbf{v} = \overrightarrow{AB}\) and \(\mathbf{w} = \overrightarrow{AD}\).

a. Let E stand for the midpoint of diagonal AC. Identify \(\overrightarrow{AE}\) in terms of \(\mathbf{v}\) and \(\mathbf{w}\).
   
   **Hint:** You’ll need to identify other arrows first, such as \(\overrightarrow{BC}\) and \(\overrightarrow{AC}\).

b. Let F stand for the midpoint of diagonal BD. Identify \(\overrightarrow{AF}\) in terms of \(\mathbf{v}\) and \(\mathbf{w}\).
   
   **Hint:** You’ll need to identify other arrows first, such as \(\overrightarrow{BD}\) and \(\overrightarrow{BF}\).

c. What have you proved about \(\overrightarrow{AE}\) and \(\overrightarrow{AF}\)? Use this to justify that E and F must be the same point.
7. This problem will lead you through proving a theorem about a midsegment of a trapezoid. Consider trapezoid ABCD with AD \parallel BC, and midpoints E and F as shown.

Let \( u = \overrightarrow{AB} \) and \( v = \overrightarrow{BC} \).

a. From your prior knowledge and/or an educated guess, what facts could we aim to prove about midsegment EF in relation to the trapezoid? (There are three things to say here: two about parallelism and one about length.)

b. Explain why \( \overrightarrow{AD} \) must equal \( kv \) for some number \( k \).

c. Identify each of the following arrows in terms of \( u \), \( v \), and/or \( k \).

\[
\begin{align*}
\overrightarrow{BA} \\
\overrightarrow{CB} \\
\overrightarrow{CD} & \quad (Hint: \overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AD}) \\
\overrightarrow{EB} \\
\overrightarrow{CF} \\
\overrightarrow{EF} & \quad (Hint: \overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CF})
\end{align*}
\]

d. Show that \( \overrightarrow{EF} \) equals a scalar multiple of \( v \).

e. Justify the intended conclusions concerning parallelism.

f. More challenging; try your best: Justify the intended conclusion concerning length.